Published by Institute of Physics Publishing for SISSA/ISAS

Received: *January 23, 2002*
Revised: *March 13, 2002*
Accepted: *March 26, 2002*

Ratio of tensions from vacuum string field theory

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Abstract: We show analytically that the ratio of the norm of sliver states agrees with the ratio of D-brane tensions. We find that the correct ratio appears as a twist anomaly.

Keywords: D-branes, Bosonic Strings, Anomalies in Field and String Theories.

 http://jhep.sissa.it/archive/papers/jhep032002050/jhep032002050.pdf

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1. Introduction

Vacuum String Field Theory (VSFT) is proposed [1] as a bosonic open string field theory around the closed string vacuum. (See [2]–[23] for related papers.) One of the interesting problems in VSFT is to reconstruct the unstable vacua describing D-branes as classical solutions of VSFT and extract the physical quantities such as the ratio of tensions [2, 3] and the ratio of the potential height to the tension [10, 14, 19].

Since the kinetic operator of VSFT is supposed to be purely ghost, the equation of motion

$$Q\Psi + \Psi \star \Psi = 0 \tag{1.1}$$

has a matter-ghost factorized solution [2]:

$$\Psi = \Psi_m \otimes \Psi_g\,, \tag{1.2}$$

where the matter part Ψ_m is given by a projection

$$\Psi_m \star_m \Psi_m = \Psi_m\,, \qquad Q\Psi_g + \Psi_g \star_g \Psi_g = 0\,. \tag{1.3}$$

Assuming that the ghost part Ψ_g are common to all the solutions, the ratio of the action is given by the ratio of the norm of projections:

$$\frac{S[\Psi']}{S[\Psi]} = \frac{\langle \Psi'_m|\Psi'_m\rangle}{\langle \Psi_m|\Psi_m\rangle}\,. \tag{1.4}$$

In [2], Rastelli, Sen and Zwiebach obtained a closed form for this ratio and checked numerically that this ratio reproduces the ratio of D-brane tensions. In this paper, we calculate this ratio analytically in the oscillator representation of string fields (see [6] for the computation of this ratio from the BCFT approach).

Naively, the norm of projection is vanishing [2, 18, 22], so the ratio (1.4) has an ill-defined expression 0/0. By carefully regularizing the ratio of determinants appearing in the norm of projection, we find that the correct ratio appears as a twist anomaly as discussed in [14].

This paper is organized as follows: in section 2, we review the conjecture of [2]. In section 3, we summarize the properties of Neumann coefficients which are used in the calculation. In section 4, we calculate the ratio of the norm of projections and show that this agrees with the expected ratio of D-brane tensions. Section 5 is devoted to discussions. In appendix A, we give a derivation of $\langle k|v_e\rangle$ and $\langle k|v_o\rangle$. Appendix B discusses the integral representation of the identities obtained in [14].

2. RSZ Conjecture

In this section, we review the statement of the conjecture in [2]. Since the 3-string vertex has a factorized form with respect to the spacetime directions, here we focus on one spatial direction. The matter part of the 3-string vertex in the $\alpha'=1$ convention is written as [24, 2]

$$\begin{aligned}|V_3\rangle &= \int\prod_{r=1}^{3}dp_{(r)}\delta\left(\sum_{s=1}^{3}p_{(s)}\right)\exp\left(\sum_{r,s=1}^{3}-\frac{1}{2}\sum_{n,m=1}^{\infty}a_n^{(r)\dagger}V_{nm}^{rs}a_m^{(s)\dagger}-\sum_{n=1}^{\infty}p_{(r)}V_{0n}^{rs}a_n^{(s)\dagger}-\right.\\ &\qquad\left.-\frac{1}{2}\sum_{r=1}^{3}p_{(r)}V_{00}^{rr}p_{(r)}\right)|0,p\rangle_{123}\\ &= \frac{(2\pi b^3)^{1/4}}{\sqrt{3}\,(V_{00}^{rr}+b/2)}\exp\left(-\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m=0}^{\infty}a_n^{(r)\dagger}V_{nm}^{\prime rs}a_m^{(s)\dagger}\right)|\Omega_b\rangle_{123}\,. \end{aligned}\tag{2.1}$$

Here $|\Omega_b\rangle$ is the vacuum for zero-mode oscillator a_0 defined by

$$a_0|\Omega_b\rangle=0\,,\qquad a_0=\frac{\sqrt{b}}{2}\widehat{p}-\frac{i}{\sqrt{b}}\widehat{x}\,,\tag{2.2}$$

where b is a numerical parameter. We can construct two types of projections: one is the sliver $|\Xi\rangle$ [25, 2] and the other is the projection $|\Xi'\rangle$ describing the lump solution. The explicit forms of $|\Xi\rangle$ and $|\Xi'\rangle$ are given by

$$\begin{aligned}|\Xi\rangle &= \det(1-M)^{1/2}(1+T)^{1/2}\exp\left(-\frac{1}{2}a^\dagger CTa^\dagger\right)|0\rangle\,,\\ |\Xi'\rangle &= \frac{\sqrt{3}\,(V_{00}^{rr}+b/2)}{(2\pi b^3)^{1/4}}\det(1-M')^{1/2}(1+T')^{1/2}\exp\left(-\frac{1}{2}a^\dagger C'T'a^\dagger\right)|\Omega_b\rangle\,.\end{aligned}\tag{2.3}$$

The various matrices appearing in $|\Xi\rangle$ and $|\Xi'\rangle$ are defined by

$$\begin{aligned} M &= CV^{11}\,, \qquad M' = C'V'^{11}\,, \\ C_{nm} &= (-1)^n\delta_{nm} \quad (n,m\geq 1)\,, \qquad C'_{nm} = (-1)^n\delta_{nm} \quad (n,m\geq 0)\,, \\ T &= \frac{1}{2M}\left(1+M-\sqrt{(1-M)(1+3M)}\right)\,, \\ T' &= \frac{1}{2M'}\left(1+M'-\sqrt{(1-M')(1+3M')}\right)\,. \end{aligned} \tag{2.4}$$

Note that the indices of primed matrices run from 0 to ∞, whereas the indices of unprimed matrices run from 1 to ∞. C is called a twist matrix, and M commutes with C.

In [2], Rastelli, Sen and Zwiebach conjectured that the ratio R of the norm of $|\Xi\rangle$ and $|\Xi'\rangle$ given by

$$\begin{aligned} R &= \frac{\langle 0|0\rangle}{\langle\Xi|\Xi\rangle}\frac{\langle\Xi'|\Xi'\rangle}{\langle\Omega_b|\Omega_b\rangle} \\ &= \frac{3\,(V_{00}^{rr}+b/2)^2}{\sqrt{2\pi b^3}}\frac{\det(1-M')^{3/4}(1+3M')^{1/4}}{\det(1-M)^{3/4}(1+3M)^{1/4}} \end{aligned} \tag{2.5}$$

reproduces the ratio of D-brane tensions, i.e.,

$$R = \frac{T_p}{2\pi\sqrt{\alpha'}T_{p+1}} = 1\,. \tag{2.6}$$

They checked this conjecture numerically by using the level truncation. In section 4, we prove (2.6) analytically.

3. Properties of Neumann coefficients

In this section, we summarize some properties of Neumann coefficients. To calculate R, first we have to know the relation between M and M', which is summarized in [2, appendix B]. M' is given by

$$M' = \begin{pmatrix} M'_{00} & M'_{0m} \\ M'_{n0} & M'_{nm} \end{pmatrix} = \begin{pmatrix} 1-\frac{2}{3}\frac{b}{V_{00}^{rr}+b/2} & -\frac{2}{3}\frac{\sqrt{2b}}{V_{00}^{rr}+b/2}\langle v_e| \\ -\frac{2}{3}\frac{\sqrt{2b}}{V_{00}^{rr}+b/2}|v_e\rangle & M+\frac{4}{3}\frac{1}{V_{00}^{rr}+b/2}(-|v_e\rangle\langle v_e|+|v_o\rangle\langle v_o|) \end{pmatrix} \tag{3.1}$$

Following [22], we use a bracket notation, such as $|v_e\rangle$, for infinite-dimensional vectors with index running from 1 to ∞. Note that bra and ket are related by the transpose, not by the hermitian conjugation. $|v_e\rangle$ and $|v_o\rangle$ in 3.1 are defined by[1]

$$|v_e\rangle = E^{-1}|A_e\rangle\,, \qquad |v_o\rangle = E^{-1}|A_o\rangle\,, \tag{3.2}$$

where $E_{nm} = \sqrt{n}\delta_{nm}$ and

$$(A_e)_n = \frac{1+(-1)^n}{2}A_n\,, \qquad (A_o)_n = \frac{1-(-1)^n}{2}A_n\,. \tag{3.3}$$

[1] W_n in [2] is related to v_e and v_o by $W_n = -\sqrt{2}(v_e+iv_o)_n$.

A_n is defined by

$$\sum_{n=\text{even}} A_n z^n + i \sum_{n=\text{odd}} A_n z^n = \left(\frac{1+iz}{1-iz}\right)^{1/3} = \exp\left(\frac{2i}{3}\tan^{-1} z\right). \tag{3.4}$$

Another information we need is the spectrum of the matrix M. This was recently obtained in [20]. The eigenvector $|k\rangle$ of M with eigenvalue $M(k)$ is defined by

$$M|k\rangle = M(k)|k\rangle\,. \tag{3.5}$$

$M(k)$ is given by

$$M(k) = -\frac{1}{2\cosh\frac{\pi k}{2}+1}\,, \tag{3.6}$$

and $|k\rangle$ is given implicitly by the generating function

$$\langle z|E^{-1}|k\rangle = \frac{1}{k}(1-e^{-k\tan^{-1} z})\,, \tag{3.7}$$

where $\langle z| = (z, z^2, \ldots)$. The inner product between two eigenvectors is proportional to the δ-function [22]:

$$\langle k|k'\rangle = \mathcal{N}(k)\delta(k-k')\,, \tag{3.8}$$

where $\mathcal{N}(k)$ is given by

$$\mathcal{N}(k) = \frac{2}{k}\sinh\frac{\pi k}{2}\,. \tag{3.9}$$

We can write down the completeness relation as

$$\mathbf{1} = \int_{-\infty}^{\infty} \frac{dk}{\mathcal{N}(k)}|k\rangle\langle k|\,. \tag{3.10}$$

We also need to know the overlap between $\langle k|$ and $|v_e\rangle, |v_o\rangle$. $\langle k|v_e\rangle$ and $\langle k|v_o\rangle$ are calculated as

$$\langle k|v_e\rangle = \frac{1}{k}\cdot\frac{\cosh\frac{\pi k}{2}-1}{2\cosh\frac{\pi k}{2}+1}\,, \qquad \langle k|v_o\rangle = \frac{\sqrt{3}}{k}\cdot\frac{\sinh\frac{\pi k}{2}}{2\cosh\frac{\pi k}{2}+1}\,. \tag{3.11}$$

See appendix A for the derivation of these equations.

The following identities obtained by Hata and Moriyama[2] [14] play an essential role in the calculation of R:

$$\langle v_e|\frac{1}{1+3M}|v_e\rangle = \frac{1}{4}V_{00}^{rr}\,, \tag{3.13}$$

$$\langle v_o|\frac{1}{1-M}|v_o\rangle = \frac{3}{4}V_{00}^{rr}\,. \tag{3.14}$$

Although $1+3M$ has a kernel $|k=0\rangle$ which is twist-odd, 3.13 is well-defined since $1+3M$ is invertible on the twist-even subspace.

In appendix B, we present the integral representations of the left-hand side of (3.13) and (3.14).

[2]Note that $\mathbf{v}_0$, $\mathbf{v}_1$ and V_{00} in [14] are related to our $|v_e\rangle$, $|v_o\rangle$ and V_{00}^{rr} by

$$\mathbf{v}_0 = -\frac{2}{3}|v_e\rangle, \quad \mathbf{v}_1 = \frac{2}{\sqrt{3}}|v_o\rangle, \quad V_{00} = \frac{1}{2}V_{00}^{rr}\,. \tag{3.12}$$

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4. Calculation of ratio

In this section, we calculate the ratio R (2.5) using the information in the previous section 3. Since $1+3M$ and $1+3M'$ have kernel, we have to regularize the ratio of their determinants. We find that the nontrivial result appears as a twist anomaly as discussed in [14].

4.1 $\det(1-M')/\det(1-M)$

Let us first consider the ratio $\det(1-M')/\det(1-M)$. From the explicit form of M' in (3.1), $\det(1-M')$ can be written as

$$\det(1-M') = \frac{2}{3}\frac{b}{V_{00}^{rr}+b/2}\det\left(1-M-\frac{4}{3}\frac{1}{V_{00}^{rr}+b/2}|v_o\rangle\langle v_o|\right). \tag{4.1}$$

Here we have used the formula

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D-CA^{-1}B). \tag{4.2}$$

By making use of the formula

$$\det(1+|u\rangle\langle v|) = 1+\langle v|u\rangle \tag{4.3}$$

and the identity (3.14), the ratio $\det(1-M')/\det(1-M)$ turns out to be

$$\begin{aligned}\frac{\det(1-M')}{\det(1-M)} &= \frac{2}{3}\frac{b}{V_{00}^{rr}+b/2}\det\left(1-\frac{4}{3}\frac{1}{V_{00}^{rr}+b/2}\frac{1}{1-M}|v_o\rangle\langle v_o|\right) \\ &= \frac{2}{3}\frac{b}{V_{00}^{rr}+b/2}\left[1-\frac{4}{3}\frac{1}{V_{00}^{rr}+b/2}\left\langle v_o\left|\frac{1}{1-M}\right|v_o\right\rangle\right] \\ &= \frac{b^2}{3\,(V_{00}^{rr}+b/2)^2}\,. \end{aligned} \tag{4.4}$$

4.2 $\det(1+3M')/\det(1+3M)$

Next we consider the ratio $\det(1+3M')/\det(1+3M)$. Since $1+3M$ and $1+3M'$ have non-zero kernel, we have to regularize the ratio of their determinants. The kernel of $1+3M$ is $|k=0\rangle$, and that of $1+3M'$ is given by

$$\begin{pmatrix} 1 \\ \frac{8}{\sqrt{2b}}\frac{1}{1+3M}|v_e\rangle \end{pmatrix}. \tag{4.5}$$

We regularize the ratio $\det(1+3M')/\det(1+3M)$ as

$$\lim_{\epsilon\to 0}\frac{\det(1+3M'+3\epsilon^2\widehat{L})}{\det(1+3M+3\epsilon^2 L)} \tag{4.6}$$

with

$$\widehat{L} = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix}. \tag{4.7}$$

Here $L=f(M)$ is an arbitrary function of M satisfying

$$L(k)>0\,,\qquad \forall k\in(-\infty,\infty)\,. \tag{4.8}$$

This condition ensures that the matrix $1+3M+3\epsilon^2 L$ has no kernel. Note that this regularization (4.6) is equivalent to the replacement

$$M \to M + \epsilon^2 L\,. \tag{4.9}$$

Using the formula (4.2) as in the previous subsection, the ratio of these determinants is written as

$$\begin{aligned}\frac{\det(1+3M'+3\epsilon^2\widehat{L})}{\det(1+3M+3\epsilon^2 L)} &= \\ &= \frac{4V_{00}^{rr}}{V_{00}^{rr}+b/2}\det\left[1+\frac{1}{1+3M+3\epsilon^2 L}\left(-\frac{4}{V_{00}^{rr}}|v_e\rangle\langle v_e| + \frac{4}{V_{00}^{rr}+b/2}|v_o\rangle\langle v_o|\right)\right].\end{aligned} \tag{4.10}$$

Since twist-even and twist-odd vectors are orthogonal to each other, the following matrix element vanishes:

$$\langle v_e|\frac{1}{1+3M+3\epsilon^2 L}|v_o\rangle = 0\,. \tag{4.11}$$

Thus, the matrix inside of the determinant in (4.10) can be rewritten as

$$\begin{aligned}&1+\frac{1}{1+3M+3\epsilon^2 L}\left(-\frac{4}{V_{00}^{rr}}|v_e\rangle\langle v_e| + \frac{4}{V_{00}^{rr}+\frac{b}{2}}|v_o\rangle\langle v_o|\right) = \\ &= \left(1-\frac{4}{V_{00}^{rr}}\frac{1}{1+3M+3\epsilon^2 L}|v_e\rangle\langle v_e|\right)\left(1+\frac{4}{V_{00}^{rr}+\frac{b}{2}}\frac{1}{1+3M+3\epsilon^2 L}|v_o\rangle\langle v_o|\right).\end{aligned} \tag{4.12}$$

Using the formula (4.3), the ratio (4.10) can be further simplified as

$$\begin{aligned}\frac{\det(1+3M'+3\epsilon^2\widehat{L})}{\det(1+3M+3\epsilon^2 L)} &= \\ &= \frac{4^2}{V_{00}^{rr}+b/2}\left[\frac{1}{4}V_{00}^{rr} - \langle v_e|\frac{1}{1+3M+3\epsilon^2 L}|v_e\rangle\right]\times \\ &\times\left[1+\frac{4}{V_{00}^{rr}+b/2}\langle v_o|\frac{1}{1+3M+3\epsilon^2 L}|v_o\rangle\right].\end{aligned} \tag{4.13}$$

From the identity (3.13), one can see that the second factor in (4.13) is vanishing in the limit $\epsilon \to 0$. On the other hand, the last factor in (4.13) is divergent since $1+3M$ is not invertible on the twist-odd subspace. However, as we will show below, the product of these two factors has a well-defined limit when $\epsilon \to 0$.

From (3.13), the second factor in (4.13) can be rewritten as

$$\frac{1}{4}V_{00}^{rr} - \langle v_e|\frac{1}{1+3M+3\epsilon^2 L}|v_e\rangle = \langle v_e|\frac{3\epsilon^2 L}{(1+3M+3\epsilon^2 L)(1+3M)}|v_e\rangle. \tag{4.14}$$

Therefore, (4.13) becomes

$$\begin{aligned}&\frac{\det(1+3M'+3\epsilon^2\widehat{L})}{\det(1+3M+3\epsilon^2 L)} = \\ &= \frac{4^2}{V_{00}^{rr}+b/2}\langle v_e|\frac{3\epsilon^2 L}{(1+3M+3\epsilon^2 L)(1+3M)}|v_e\rangle\left[1+\frac{4}{V_{00}^{rr}+b/2}\langle v_o|\frac{1}{1+3M+3\epsilon^2 L}|v_o\rangle\right] \\ &= \frac{4^2}{V_{00}^{rr}+b/2}\langle v_e|\frac{3\epsilon L}{(1+3M+3\epsilon^2 L)(1+3M)}|v_e\rangle\left[\epsilon+\frac{4}{V_{00}^{rr}+b/2}\langle v_o|\frac{\epsilon}{1+3M+3\epsilon^2 L}|v_o\rangle\right].\end{aligned} \tag{4.15}$$

Now we can take the limit $\epsilon \to 0$. For definiteness, we only consider the limit $\epsilon \to 0^+$, i.e., ϵ approaches 0 from above, but we can show that the $\epsilon \to 0^-$ limit gives the same answer. Using the formula

$$\lim_{\epsilon\to 0^+} \frac{\epsilon}{\epsilon^2 + x^2} = \pi\delta(x)\,, \tag{4.16}$$

we can see that the eigenvalue of the matrix $\epsilon/(1+3M+3\epsilon^2 L)$ appearing in (4.15) approaches the δ-function of k:

$$\lim_{\epsilon\to 0^+} \frac{\epsilon}{1+3M(k)+3\epsilon^2 L(k)} = \frac{2}{\sqrt{L(0)}}\delta(k)\,. \tag{4.17}$$

Therefore, the following type of matrix element is given by the value at $k=0$:

$$\begin{aligned}\lim_{\epsilon\to 0^+} \langle v|\frac{\epsilon}{1+3M+3\epsilon^2 L}X|v\rangle &= \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \frac{dk}{\mathcal{N}(k)} \frac{\epsilon}{1+3M(k)+3\epsilon^2 L(k)} X(k)\langle k|v\rangle^2 \\ &= \int_{-\infty}^{\infty} \frac{dk}{\mathcal{N}(k)} \frac{2}{\sqrt{L(0)}}\delta(k)X(k)\langle k|v\rangle^2 \\ &= \frac{2}{\sqrt{L(0)}}\frac{X(k)}{\mathcal{N}(k)}\langle k|v\rangle^2\Big|_{k=0}.\end{aligned} \tag{4.18}$$

Here we have used the completeness relation (3.10) of $|k\rangle$. This kind of phenomenon that the nontrivial contribution comes only from $k=0$ is called twist anomaly in [14]. Substituting the eigenvalue $M(k)$ (3.6) and $\langle k|v_{e,o}\rangle$ (3.11) into the general formula (4.18), and using $\mathcal{N}(0)=\pi$ (3.9), the limit of the matrix elements appearing in (4.15) can be evaluated as

$$\begin{aligned}\lim_{\epsilon\to 0^+} \langle v_e|\frac{3\epsilon L}{(1+3M+3\epsilon^2 L)(1+3M)}|v_e\rangle &= \frac{\pi}{8}\sqrt{L(0)}\,, \\ \lim_{\epsilon\to 0^+} \langle v_o|\frac{\epsilon}{1+3M+3\epsilon^2 L}|v_o\rangle &= \frac{\pi}{6}\frac{1}{\sqrt{L(0)}}\,.\end{aligned} \tag{4.19}$$

Finally, the ratio $\det(1+3M')/\det(1+3M)$ is found to be

$$\frac{\det(1+3M')}{\det(1+3M)} \equiv \lim_{\epsilon\to 0}\frac{\det(1+3M'+3\epsilon^2\widehat{L})}{\det(1+3M+3\epsilon^2 L)} = \frac{4\pi^2}{3\left(V_{00}^{rr}+b/2\right)^2}\,. \tag{4.20}$$

4.3 Proof of $R=1$

Plugging the ratios (4.4) and (4.20) into the definition of R (2.5), we arrive at the final result:

$$R = \frac{3\left(V_{00}^{rr}+b/2\right)^2}{\sqrt{2\pi b^3}}\left[\frac{b^2}{3\left(V_{00}^{rr}+b/2\right)^2}\right]^{3/4}\left[\frac{4\pi^2}{3\left(V_{00}^{rr}+b/2\right)^2}\right]^{1/4} = 1\,. \tag{4.21}$$

Namely, the ratio of D-brane tensions is correctly reproduced from the ratio of the norm of projections! Note that the result is independent of the parameter b as conjectured in [2].

5. Discussions

In this paper, we showed analytically that the ratio of the norm of projections correctly reproduces the ratio of D-brane tensions.

Here we would like to discuss on the issue of the regularization dependence of R. We showed that the regularization written as a shift of M (4.9) gives the correct ratio for arbitrary choice of L. Instead of shifting M, we can consider some different regularizations, e.g., we can cut-off the dangerous region $k \sim 0$ as

$$\int_{-\infty}^{\infty} dk \to \int_{|k|>\epsilon} dk \,. \tag{5.1}$$

Unfortunately, this method leads to a wrong result:

$$\begin{aligned} \frac{1}{4}V_{00}^{rr} - \langle v_e|\frac{1}{1+3M}|v_e\rangle &\sim \frac{1}{4}V_{00}^{rr} - \int_{|k|>\epsilon} \frac{dk}{\mathcal{N}(k)} \frac{\langle k|v_e\rangle^2}{1+3M(k)} \sim \frac{\pi}{24}\epsilon \,, \\ \langle v_o|\frac{1}{1+3M}|v_o\rangle &\sim \int_{|k|>\epsilon} \frac{dk}{\mathcal{N}(k)} \frac{\langle k|v_o\rangle^2}{1+3M(k)} \sim \frac{2}{\pi\epsilon} \,, \\ \Longrightarrow \frac{\det(1+3M')}{\det(1+3M)} &= \frac{4^2}{3\,(V_{00}^{rr}+b/2)^2} . \end{aligned} \tag{5.2}$$

Therefore, the ratio R seems to be dependent on the regularization.

What is wrong of this regularization and why our regularization give the correct result? In the following, we will present a plausible (but not conclusive) argument on this issue. One obvious bad point of this cut-off regularization is that the cut-off procedure does not commute with the matrix multiplication:

$$(P_\epsilon X P_\epsilon)(P_\epsilon Y P_\epsilon) \neq P_\epsilon XY P_\epsilon \,, \tag{5.3}$$

where P_ϵ is the projection defined by

$$P_\epsilon = \int_{|k|>\epsilon} \frac{dk}{\mathcal{N}(k)} |k\rangle\langle k| \,. \tag{5.4}$$

For example, $(P'_\epsilon M' P'_\epsilon)^2 \neq P'_\epsilon M'^2 P'_\epsilon$ where $P'_\epsilon = \mathrm{diag}(1, P_\epsilon)$. In consequence, the associativity of the star product is broken explicitly under this regularization, and hence the gauge symmetry of VSFT is not preserved. On the other hand, one can show that our regularization (4.9) (at least formally) respects the associativity of the star product if one changes the Neumann matrices $M^{rs} \to M_\epsilon^{rs}$ as

$$M_\epsilon^{11} = M + \epsilon^2 L \,, \qquad M_\epsilon^{12} + M_\epsilon^{21} = 1 - M_\epsilon^{11} \,, \qquad M_\epsilon^{12} M_\epsilon^{21} = (M_\epsilon^{11})^2 - M_\epsilon^{11} \,, \tag{5.5}$$

and similarly for M'^{rs}.

This is reminiscent of the situation in Yang-Mills theories. In that case, we have to choose a regularization respecting the gauge symmetry, e.g., the dimensional regularization. On the other hand, the Pauli-Villars regularization would break the gauge invariance and lead to a wrong answer.

In our case, the guiding principle of regularization is the *gauge symmetry of VSFT*. Note that R is a gauge invariant quantity since it is written as a ratio of the classical action of lump to that of sliver. Therefore, one can expect that all the regularizations respecting the gauge symmetry will give the same result, namely $R = 1$.

Acknowledgments

I would like to thank David Kutasov for discussion.

A. Calculation of $\langle k|v_e\rangle$ and $\langle k|v_e\rangle$

In this appendix, we give a derivation of (3.11). From (3.4), the generating functions of $|A_e\rangle$ and $|A_o\rangle$ are found to be

$$\langle z|A_e\rangle = \cosh\left(\frac{2i}{3}\tan^{-1} z\right) - 1\,, \qquad \langle z|A_o\rangle = -i\sinh\left(\frac{2i}{3}\tan^{-1} z\right). \tag{A.1}$$

Let us first consider $\langle k|v_e\rangle$. This can be extracted from the generating function (A.1) as

$$\begin{aligned}\langle k|v_e\rangle &= \langle k|E^{-1}|A_e\rangle = \int_{-\pi/2}^{3\pi/2}\frac{d\theta}{2\pi}\langle k|E^{-1}|e^{i\theta}\rangle\langle e^{-i\theta}|A_e\rangle \\ &= \int_{-\pi/2}^{3\pi/2}\frac{d\theta}{2\pi}\frac{1}{k}(1-e^{-k\tan^{-1}e^{i\theta}})\cosh\left(\frac{2i}{3}\tan^{-1}e^{-i\theta}\right).\end{aligned} \tag{A.2}$$

Note that the constant term in $\langle z|A_e\rangle$ can be neglected since $\langle k|$ does not have $n = 0$ component. As discussed in [22], this integral can be evaluated by making a change of integration variable from θ to x as (see [22] for detail)

$$\begin{aligned}\tan^{-1} e^{i\theta} &= \frac{\pi}{4} + ix \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right) \\ &= -\frac{\pi}{4} - ix \qquad \left(\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right).\end{aligned} \tag{A.3}$$

Then (A.2) is rewritten as

$$\begin{aligned}\langle k|v_e\rangle &= \frac{2}{\pi k}\int_{-\infty}^{\infty}\frac{dx}{\cosh 2x}\left[1-\cosh\left(\frac{\pi k}{4}+ikx\right)\right]\cosh\left(\frac{2x}{3}+\frac{\pi i}{6}\right) \\ &= \frac{2}{\pi k}\int_0^{\infty}\frac{dx}{\cosh 2x}\Bigg[2\cosh\frac{2x}{3}\cos\frac{\pi}{6} - \cosh\left(\frac{2x}{3}+ikx\right)\cosh\left(\frac{\pi k}{4}+\frac{\pi i}{6}\right) - \\ &\qquad\qquad - \cosh\left(\frac{2x}{3}-ikx\right)\cosh\left(\frac{\pi k}{4}-\frac{\pi i}{6}\right)\Bigg] \\ &= \frac{1}{2k}\left[2 - \frac{\cosh\left(\frac{\pi k}{4}+\frac{\pi i}{6}\right)}{\cosh\left(\frac{\pi k}{4}-\frac{\pi i}{6}\right)} - \frac{\cosh\left(\frac{\pi k}{4}-\frac{\pi i}{6}\right)}{\cosh\left(\frac{\pi k}{4}+\frac{\pi i}{6}\right)}\right] \\ &= \frac{1}{k}\frac{\cosh\frac{\pi k}{2}-1}{2\cosh\frac{\pi k}{2}+1}.\end{aligned} \tag{A.4}$$

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Here we have used the formula

$$\int_0^\infty dx \frac{\cosh ax}{\cosh bx} = \frac{\pi}{2b} \cdot \frac{1}{\cos \frac{\pi a}{2b}}, \quad (|\mathrm{Re}\, a| < \mathrm{Re}\, b) . \tag{A.5}$$

$\langle k|v_o\rangle$ can be calculated in the same way:

$$\begin{aligned} \langle k|v_o\rangle &= -i \int_{-\pi/2}^{3\pi/2} \frac{d\theta}{2\pi} \frac{1}{k} (1 - e^{-k \tan^{-1} e^{i\theta}}) \sinh\left(\frac{2i}{3} \tan^{-1} e^{-i\theta}\right) \\ &= -\frac{2i}{\pi k} \int_{-\infty}^{\infty} \frac{dx}{\cosh 2x} \sinh\left(\frac{\pi k}{4} + ikx\right) \sinh\left(\frac{2x}{3} + \frac{\pi i}{6}\right) \\ &= -\frac{2i}{\pi k} \int_0^\infty \frac{dx}{\cosh 2x} \left[\cosh\left(\frac{2x}{3} + ikx\right) \cosh\left(\frac{\pi k}{4} + \frac{\pi i}{6}\right) - \right. \\ &\qquad\qquad \left. - \cosh\left(\frac{2x}{3} - ikx\right) \cosh\left(\frac{\pi k}{4} - \frac{\pi i}{6}\right)\right] \\ &= -\frac{i}{2k} \left[\frac{\cosh\left(\frac{\pi k}{4} + \frac{\pi i}{6}\right)}{\cosh\left(\frac{\pi k}{4} - \frac{\pi i}{6}\right)} - \frac{\cosh\left(\frac{\pi k}{4} - \frac{\pi i}{6}\right)}{\cosh\left(\frac{\pi k}{4} + \frac{\pi i}{6}\right)}\right] \\ &= \frac{\sqrt{3}}{k} \frac{\sinh \frac{\pi k}{2}}{2\cosh \frac{\pi k}{2} + 1} . \end{aligned} \tag{A.6}$$

B. Integral representation of Hata-Moriyama's identities

Using the completeness relation (3.10), the left hand sides of (3.13) and (3.14) are written as

$$\begin{aligned} \langle v_e| \frac{1}{1+3M} |v_e\rangle &= \int_{-\infty}^{\infty} \frac{dk}{\mathcal{N}(k)} \frac{1}{1+3M(k)} \langle k|v_e\rangle^2 = \frac{1}{4} I , \\ \langle v_o| \frac{1}{1-M} |v_o\rangle &= \int_{-\infty}^{\infty} \frac{dk}{\mathcal{N}(k)} \frac{1}{1-M(k)} \langle k|v_o\rangle^2 = \frac{3}{4} I , \end{aligned} \tag{B.1}$$

where I is given by

$$I = \int_{-\infty}^{\infty} \frac{dk}{k} \frac{\sinh \frac{\pi k}{2}}{\left(\cosh \frac{\pi k}{2} + 1\right)\left(2\cosh \frac{\pi k}{2} + 1\right)} = \int_{-\infty}^{\infty} \frac{dt}{t} \frac{\sinh t}{(\cosh t + 1)(2\cosh t + 1)} . \tag{B.2}$$

Therefore, the identities (3.13) and (3.14) obtained in [14] are equivalent to

$$I = V_{00}^{rr} = \log \frac{27}{16} . \tag{B.3}$$

The integral (B.3) can be evaluated by summing up the residues of the following series of poles on the upper half t-plane:[3]

$$t = \pi i + 2n\pi i , \qquad \frac{2\pi i}{3} + 2n\pi i , \qquad \frac{4\pi i}{3} + 2n\pi i , \qquad (n = 0, 1, 2, \dots) . \tag{B.4}$$

[3]This integral is computed in the recent paper [26] in a more general setup.

Then I is written as

$$I = \sum_{n=0}^{\infty} \left(-\frac{4}{2n+1} + \frac{3}{3n+1} + \frac{3}{3n+2} \right) . \tag{B.5}$$

By introducing the function

$$\begin{aligned} I(a) &= \sum_{n=0}^{\infty} \left(-\frac{4}{2n+1} + \frac{3}{3n+1} + \frac{3}{3n+2} \right) a^n \\ &= \sum_{r=0}^{1} 2a^{-1/2} (-1)^r \log \left(1 - (-1)^r a^{1/2} \right) - \\ &\quad - \sum_{r=0}^{2} a^{-2/3} e^{2\pi r i/3} \left(1 + e^{2\pi r i/3} a^{1/3} \right) \log \left(1 - e^{2\pi r i/3} a^{1/3} \right) , \end{aligned} \tag{B.6}$$

the summation (B.5) is evaluated as

$$I = \lim_{a \to 1} I(a) = \log \frac{27}{16} . \tag{B.7}$$

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